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AIZERMAN'S PROBLEM†

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Aizerman's problem is solved in the affirmative in the case when the right-hand side of the differential equation is a self-adjoint matrix.

In n-space R_n we consider a non-linear system

$$\dot{x}_1 = \sum_{k=1}^{n} a_{1k}x_k + f(x_1), \quad \dot{x}_i = \sum_{k=1}^{n} a_{ik}x_k, \quad i=2,\ldots,n \tag{1}$$

and together with it the linear system obtained from (1) when $f(x_1) = bx_1$.

Aizerman's problem [1] is as follows: if it is known that the trivial solution of the linear system is asymptotically stable for all b satisfying the condition $\alpha < b < \beta$, will the trivial solution of the non-linear system (1) be stable in the large of the following condition is satisfied

$$\alpha < f(x_1)/x_1 < \beta \tag{2}$$

This problem has inspired much research. It has been shown that condition (2) is not sufficient for stability in second-order [2] and third-order [3] systems.

This note presents a study of stability for systems of non-linear equations of a more general form than (1)

$$\dot{\mathbf{x}} = A\mathbf{x} + F(\mathbf{x}) \tag{3}$$

from which it follows that Aizerman's problem has a positive solution for self-adjoint matrices, provided that $-\infty < \alpha$, $\beta < \infty$. Here $\mathbf{x} = (x_1, \ldots, x_n)$, $A = \{a_{ik}\}$ $(i, k = 1, 2, \ldots, n)$, $F(\mathbf{x}) = (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n))$.

Our stability analysis will be carried out in the spaces R_n, E_n, where E_n is Euclidean space. We shall use the following notation:

$$R(\mathbf{a}, r) = \{\mathbf{x} \in R_n : \|\mathbf{x} - \mathbf{a}\| \leq r\}, \quad S(\mathbf{a}, r) = \{\mathbf{x} \in R_n : \|\mathbf{x} - \mathbf{a}\| = r\}$$

$$\operatorname{Re} K = K_R = (K + K^*)/2, \quad \Lambda(K) = \lim_{h \downarrow 0}(\|I + hK\| - 1)h^{-1}$$

where $\Lambda(K)$ is the logarithmic norm of the linear operator K [4], $s_j(s_j(K))$ denote the s-numbers of K, that is, the eigenvalues of the operator K^*K and $\sigma(k)$ is the spectrum of K.

Consider a system of linear equations

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{x} \tag{4}$$

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where the matrix $B=\{b_{ik}\}$ $i, k=1, 2, \ldots, n$ has been chosen so that

$$\sigma(\mathrm{Re}(A+B))\leq -\alpha,\ \ \alpha=\mathrm{const}>0 \tag{5}$$

The set of matrices B for which condition (5) holds will be denoted by G.

We will fix an arbitrary element $\mathbf{z}=(z_1, \ldots, z_n)\in R_n$ and associate with it the matrix $C(\mathbf{z})=\{c_{ik}\}$ $(i, k=1, 2, \ldots, n)$, whose elements are $c_{ik}=f_i(z_1, \ldots, z_n)\ (mz_k)^{-1}$ if $z_k\neq 0$, $c_{ik}=d_{ik}$ if $z_k=0$, where $d_{ik}=\lim_{z_k\to 0} f_i(z_1, \ldots, z_n)z_k^{-1}$ if the limit exists, $d_{ik}=0$ if the limit does not exist, and m is the number of non-zero elements z_k $(k=1, 2, \ldots, n)$ of the vector $\mathbf{z}$.

Theorem 1. Suppose that for any $\mathbf{z}\in R_n$ the matrix $C(\mathbf{z})$ is in the set G, the functions $f_i(z_1, \ldots, z_n)$ $(i=1, 2, \ldots, n)$ are continuous and $f_i(0, \ldots, 0)=0$ $(i=1, 2, \ldots, n)$. Then the solution of the system of equations (3) is stable in the large.

Proof. Suppose that at a time T the trajectory of system (3) passes through a point $\mathbf{z}\in R_n$ with norm $\|\mathbf{z}\|=r$. We will show that a time interval Δt_1 exists during which the trajectory of the solution of system (4) passes from the sphere $S(\mathbf{0},r)$ into a sphere $R(\mathbf{0}, r_1)$, where $r_1=e^{-\alpha\Delta t_1/2}r$. To do this we will represent Eq. (3) in the form

$$\begin{aligned}&\dot{\mathbf{x}}=A\mathbf{x}+C\mathbf{x}+D(\mathbf{x}) \qquad (6)\\ &D(\mathbf{x})=(d_1(x_1,\ldots,x_n),\ldots,d_n(x_1,\ldots,x_n)),\ d_i(x_1,\ldots,x_n)=\\ &=f_i(x_1,\ldots,x_n)-f_i(z_1,\ldots,z_n)-\sum_{k=1}^{n}c_{ik}(x_k-z_k)\end{aligned}$$

The solution of Eq. (6) may be written for $t\geq T$ in the form

$$\mathbf{x}(t)=e^{(A+C)(t-T)}\mathbf{x}(T)+\int_T^t e^{(A+C)(t-\tau)}D(\mathbf{x}(\tau))d\tau \tag{7}$$

Changing to norms in (7), we have

$$\|\mathbf{x}(t)\|\leq e^{-\alpha(t-T)}r+\int_T^t e^{-\alpha(t-\tau)}\|D(\mathbf{x}(\tau))\|d\tau \tag{8}$$

Let Δt_1 denote a time interval during which $\|D(\mathbf{x}(t))\|\leq\alpha/2\|\mathbf{x}(\tau)\|$. Then when $t\in[T, T+\Delta t_1]$ inequality (8) may be strengthened, replacing $D(\mathbf{x}(\tau))$ by $\alpha\mathbf{x}(\tau)/2$. Multiplying both sides of the strengthened inequality by $e^{\alpha t}$, we obtain

$$\varphi(t)\leq e^{\alpha T}r+\frac{\alpha}{2}\int_T^t\varphi(\tau)d\tau,\ \ \varphi(t)=e^{\alpha t}\|\mathbf{x}(t)\| \tag{9}$$

Applying the Gronwall–Bellman inequality to (9), we see that for $T\leq t\leq T+\Delta t_1$ we have $\|\mathbf{x}(t)\|\leq e^{\alpha(t-T)/2}r$. Consequently, for $t_1=T+\Delta t_1$ we obtain the estimate $r_1=e^{-\alpha\Delta t_1/2}r$. Continuing the process, we finally see that at times $t_2, t_3, \ldots$ the trajectory of the solution of Eq. (3) cuts the spheres $S(\mathbf{0}, r_2), S(\mathbf{0}, r_3), \ldots$.

For the radii r_k of the spheres we have

$$\begin{aligned}&r_k=r\exp[-\alpha(\Delta t_1+\ldots+\Delta t_k)/2]\\ &\Delta t_k=t_k-t_{k-1},\ \ k=1,2,\ldots,\ \ t_0=T\end{aligned}$$

We have thus shown that the trajectory of the solution to system (3), having started in a sphere $S(\mathbf{0}, r)$, will not leave that sphere. Applying Peano's theorem [6, p. 10], we see that the trajectory of system (4) can be continued to the infinite time interval $[T, \infty)$.

Let T^* be the sum of the time intervals Δt_i constructed above, beginning with Δt_1. There are two possibilities: (1) $T^* = \text{const} < \infty$, (2) $T^* = \infty$.

Consider the first possibility. We will show that $\|\mathbf{x}(T+T^*)\| = 0$. We will prove this indirectly. Suppose that $\|\mathbf{x}(T+T^*)\| = d > 0$. Then, as shown previously, a time interval $\Delta t^* \geq 0$ exists such that $\|\mathbf{x}(T+T^*+\Delta t^*)\| \leq e^{-\alpha\Delta t^*/2}\|\mathbf{x}(T+T^*)\|$. It follows from the definition of T^* that $\Delta t^* = 0$. This contradiction implies that $\|\mathbf{x}(T+T^*)\| = 0$, i.e. the solution is asymptotically stable.

Consider the second possibility. It was shown above that $\|\mathbf{x}(t_k)\| \leq \exp[-\alpha(\Delta t_1 + \ldots + \Delta t_k/2]r$ $(k = 1, 2, \ldots)$. Taking into account that $T^* = \infty$, we have $\lim \|\mathbf{x}(t_k)\| = 0$ as $k \to \infty$. Since whenever $t_k \leq t$ the point $\mathbf{x}(t)$ lies inside the sphere $S(\mathbf{0}, \|\mathbf{x}(t_k)\|)$, it follows that the solution of system (4) is indeed asymptotically stable in this case, whatever the initial approximation.

Let G^* denote the set of matrices $B = \{b_{ik}\}$ $(i, k = 1, 2, \ldots, n)$, such that $\Lambda(A+B) \leq \alpha$, $\alpha = \text{const} < 0$.

Theorem 2. Suppose that for any $\mathbf{z} \in R_n$ the matrix $C(\mathbf{z})$ is in the set G^*, the functions $f_i(z_i, \ldots, z_n)$ are continuous and $f_i(0, \ldots, 0) = 0$ $(i = 1, 2, \ldots, n)$. Then the solution of system (4) is stable in the large.

The proof is similar to that of Theorem 1. The only difference is that in passing from formula (7) to inequality (8) one uses the well-known property of the logarithmic norm: $\|e^{A+C}\| \leq e^{\Lambda(A+C)}$ (see [4]).

Let G^{**} denote the set of matrices $B = \{b_{ik}\}$ $(i, k = 1, 2, \ldots, n)$ such that $s^* = \max s(A+B) \leq \alpha$, $\alpha = \text{const} < 0$.

Theorem 3. Suppose that for any $z \in E_n$ the matrix $C(z)$ lies in the set G^{**}, the functions $f_i(z_1, \ldots, z_n)$ are continuous and $f_i(0, \ldots, 0) = 0$ $(i = 1, 2, \ldots, n)$. Then the solution of system (3) is stable in the large.

The proof is similar to that of Theorem 1. The difference is as follows. It is well known [4] that $\|e^{A+C}\| \leq e^{\|A+C\|}$. The norm $\|A+C\|$ is estimated in E_n by the following chain of inequalities

$$\|(A+C)\mathbf{x}\| = ((A+C)\mathbf{x}, (A+C)\mathbf{x})^{1/2} = (UT\mathbf{x}, UT\mathbf{x})^{1/2} =$$
$$= (T\mathbf{x}, T\mathbf{x})^{1/2} = \|T\mathbf{x}\| \leq \max_j s_j \|\mathbf{x}\|$$

where we have used the representation of the operator $A + C = UT$ as a product of a purely isometric operator U and the operator $T = (A+C)^*(A+C)$ (see [5]).

Let us return to Aizerman's problem. Let $a = \lim f(x)/x$ as $x \to 0$ if the limit exists, or $a = 0$ otherwise.

We shall assume that (1) the matrix $A = \{a_{ij}\}$ $i, j = 1, 2, \ldots n$ is self-adjoint, (2) the linear system obtained from (1) when $f(x_1) = bx_1$ is asymptotically stable for any $b \in B$, $B = [\alpha, \beta] \cup \{a\}$, (3) Condition (2) holds, and (4) $\alpha > -\infty$, $\beta < \infty$. Since the matrix A is symmetric, the same is true of the matrix $\overline{A}_b = \{\overline{a}_{lk}\}$ $(l, k = 1, 2, \ldots, n)$, where $\overline{a}_{11} = a_{11} + b$, $\overline{a}_{ij} = a_{ij}$ if $(i, j) \neq (1, 1)$.

We will show that if the solution of the linear system obtained from (1) when $f(x_1) = bx_1$, $b \in B$, is asymptotically stable, then there exists a constant $\gamma < 0$ exists such that, for all such values of b, the eigenvalues of the matrix $\overline{A}_b$ are less than γ.

We will prove this indirectly. Suppose that there is a sequence of numbers $b_k (b_k \in B)$ such that $\lim \max(\sigma(\overline{A}_{b_k})) = 0$ as $k \to \infty$. The sequence b_k contains a subsequence which converges to a number b^* such that $\alpha \leq b^* \leq \beta$. Since $\max\sigma(\overline{A}_{b^*}) \leq \gamma(b^*) < 0$, it follows [7, chap. 14] that an ε-neighbourhood $(\varepsilon = \gamma(b^*)/2)$ of b^* exists for whose points b the spectrum of the matrix $\overline{A}_b$ lies to the left of the number $\gamma(b^*) + \varepsilon \leq \gamma(b^*)/2 < 0$. This contradiction implies that $\max\sigma(\overline{A}_b) \leq \gamma$ for all $\alpha \leq b \leq \beta$.

Since the matrix $\overline{A}_b$ is self-adjoint for all $b \in B$, it follows that $\sigma(\overline{A}_b) = \sigma(\mathrm{Re}\,\overline{A}_b)$. Consequently, Theorem 1 can be applied to system (1). This leads to the following assertion.

Theorem 4. Suppose that the linear system obtained from system (1) when $f(x_1) = bx_1$ is asymptotically stable for any b such that $b \in B$, the matrix A is self-adjoint, conditions (2) hold and $\alpha > -\infty$, $\beta < \infty$. Then the non-linear system (1) is stable in the large.

Corollary 1. Suppose that the solution of the system $\mathbf{x} = 1/2(\overline{A}_b + \overline{A}_b^*)\mathbf{x}$ is asymptotically stable for any b such that $b \in [\alpha, \beta] \cup \{a\}$, condition (2) holds, and $\alpha > -\infty$, $\beta < \infty$. Then the non-linear system (1) is stable in the large. This assertion follows from Theorem 2 and the equality $\sigma(1/2(\overline{A}_b + \overline{A}_b^*)) = \sigma(\mathrm{Re}\,\overline{A}_b)$.

Corollary 2. Suppose that the solution of the system $\mathbf{x} = \overline{A}_b^* \overline{A}_b \mathbf{x}$ is asymptotically stable for all b such that $b \in B$, condition (2) holds, and $\alpha > -\infty$, $\beta < \infty$. Then the non-linear system (1) is stable in the large.

This assertion follows from Theorems 3 and 4.

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